CS 237: Probability in Computing

Wayne Snyder
Computer Science Department
Boston University

Lecture 7:

- Review: Permutations Permutations with Duplicates
- Circular Permutations
- Combinations
- Applications of combinatorics to probability
- Poker Probability (analytical results for lab problems in HW 05)

	Selection Without Replacement	Selection With Replacement
Ordered Outcome (Sequence or String)	Permutations	Enumerations
Unordered Outcome (Set or Multiset)	Combinations	(We will not study this possibility...)

Finite Combinatorics

K-Permutations

If we do not simply rearrange all N objects, but consider selecting K <= N of them, and arranging these K, we have a K-Permutation indicated by P(N,K).

Canonical Problem 1(b): Suppose you have N students S_1, S_2, ..., S_n. In how many ways can K of them be arranged in a sequence in K chairs?

Formula:

$$P(N,K) = \overbrace{N * (N-1) * \cdots * (N-K+1)}^{\text{K terms}} = \frac{N * (N-1) * \cdots * (N-K+1) * (N-K) * \cdots * 1}{(N-K) * \cdots * 1} = \frac{N!}{(N-K)!}$$

Example: How many passwords of 8 lower-case letters and digits can be made, if you are not allowed to repeat a letter or a digit?

Answer: The "not allowed to repeat" means essentially that you are doing this "without replacement." So we have P(36,8) = 36! / 28! = 1,220,096,908,800.

Note: The usual formula at the extreme right is extremely inefficient. The first formula is the most efficient, if not the shortest to write down!

$$P(N,K) = N * (N-1) * \cdots * (N-K+1)$$

Finite Combinatorics

	Selection Without Replacement	Selection With Replacement
Ordered Outcome (Sequence or String)	Permutations	Enumerations
Unordered Outcome (Set or Multiset)	Combinations	(We will not study this possibility...)

Example: Departments in CAS are abbreviated by two capital letters. How many possible department abbreviations are there?

Finite Combinatorics

	Selection Without Replacement	Selection With Replacement
Ordered Outcome (Sequence or String)	Permutations	Enumerations
Unordered Outcome (Set or Multiset)	Combinations	(We will not study this possibility...)

Permutations with Repetitions

If you have N (non-distinct) elements, consisting of m (distinct) elements with multiplicities K_1, K_2, ..., K_m, that is, $K_1 + K_2 + \ldots + K_m = N$, then the number of distinct permutations of the N elements is

$$\frac{N!}{K_1! * K_2! * \cdots K_m!}$$

Example: How many distinct (different looking) permutations of the word "MISSISSIPPI" are there?

Solution: There are 11 letters, with multiplicities:

M: 1
I: 4
S: 4
P: 2

Therefore the answer is $$\frac{11!}{1! * 4! * 4! * 2!} = \frac{39,916,800}{1 * 24 * 24 * 2} = 34,650$$

Finite Combinatorics

	Selection Without Replacement	Selection With Replacement
Ordered Outcome (Sequence or String)	Permutations	Enumerations
Unordered Outcome (Set or Multiset)	Combinations	(We will not study this possibility...)

Example: How many distinct (different looking) permutations of the word "SCIENCE" are there?

Finite Combinatorics

	Selection Without Replacement	Selection With Replacement
Ordered Outcome (Sequence or String)	Permutations	Enumerations
Unordered Outcome (Set or Multiset)	Combinations	(We will not study this possibility...)

Example: How many distinct (different looking) permutations of the word "SCIENCE" are there?

Solution: There are 7 letters, with multiplicities:

S, I, N: 1

C: 2

E: 2

Therefore the answer is $$\frac{7!}{2! * 2!} = \frac{5040}{4} = 1260$$

Finite Combinatorics

	Selection Without Replacement	Selection With Replacement
Ordered Outcome (Sequence or String)	Permutations	Enumerations
Unordered Outcome (Set or Multiset)	Combinations	(We will not study this possibility...)

Circular Permutations

A related idea is permutations of elements arranged in a circle. The issue here is that (by the physical arrangement in a circle) we do not care about the exact position of each elements, but only "who is next to whom." Therefore, we have to correct for the overcounting by dividing by the number of possible rotations around the circle.

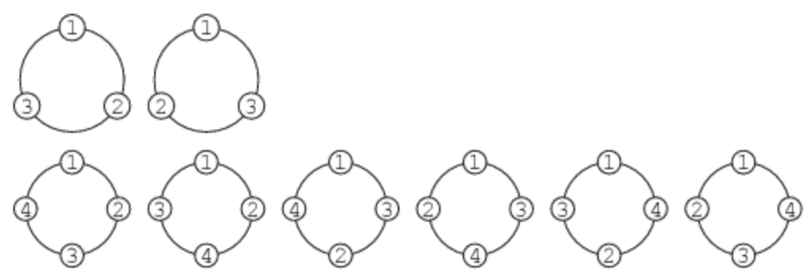
Example: There are 6 guests to be seated at a circular table. How many arrangements of the guests are there?

Hint: The idea here is that if everyone moved to the left one seat, the arrangement would be the same; it only matters who is sitting next to whom. So we must factor out the rotations. For N guests, there are N rotations of every permutation.

Solution: There are 6! permutations of the guests, but for any permutation, there are 6 others in which the same guests sit next to the same people, just in different rotations.

Formula: There are $\dfrac{N!}{N} = (N-1)!$

circular permutations of N distinct objects.



Finite Combinatorics

Application of Enumerations and Permutations

The Birthday Problem: What is the probability that at least two students in a class of size K have the same birthday? Assume all birthdays are equally likely throughout the year and each year has 365 days.

Our class has 160 students. What is the probability that two people in the class have the same birthday?

Finite Combinatorics

Application of Enumerations and Permutations

	Selection Without Replacement	Selection With Replacement
Ordered Outcome (Sequence or String)	Permutations	Enumerations
Unordered Outcome (Set or Multiset)	Combinations	(We will not study this possibility...)

The Birthday Problem: What is the probability that at least two students in a class of size K have the same birthday? Assume all birthdays are equally likely throughout the year and each year has 365 days.

Solution: There are 365 possibilities for each student. Thus, the sample space has 365^K points (it is an enumeration!). The number of ways that no two students share a birthday is P(365,K) (it is a K-permutation).

Using the inverse method, we compute $$1.0 - \frac{P(365, K)}{365^K}$$

For K = 160 (our class), we have

$$1.0 - \frac{\begin{array}{l}9234832421707412614680608557736305117109516728608790396126358309295718207691899163819591642990947481873894197 29\\1925225393513737065645321545715073505682350177621789832535800255420674482699788256349336430935628443734521432 04\\6921609957219625735194298716138698322521663568904038132081914572821552138194842574689782995050879242053163928 19\\86851880034845589504000000000000000000000000000000000000000\end{array}}{\begin{array}{l}9265274979057819523173770035807034347522181522415489971507141672989877531635267825016575086407764581963130893 45\\1347662627875208297986305003794328992247648181111249087585013596071688448197331101048207039652641995044345881 73\\6269037620776262285608772658661581381317568876941308815420457587548421517163024625195830913774885029944120745 93\\537873703082916496646205121711514654872510199368207395309582352638244628906 25\end{array}}$$

$$= 0.99999999999999999003285661506994214932465056402926046872288053 9$$

Finite Combinatorics

	Selection Without Replacement	Selection With Replacement
Ordered Outcome (Sequence or String)	Permutations	Enumerations
Unordered Outcome (Set or Multiset)	Combinations	(We will not study this possibility...)

Combinations

When we are selecting without replacement and creating a set, we have a combination.

Canonical Problem: Suppose you have N people and want to choose a committee of K people. How many possible choices are there?

How is this different from a K-Permutation? Suppose you have N people; how many ways of choosing a sequence of K people from these N?

$$P(N,K) = \frac{N!}{(N-K)!}$$

The difference is between a sequence (K-permutation) and a set (combination), so we have to apply the **unordering principle** and divide by the number of permutations of K people, or K!.

Formula:

$$\binom{N}{K} = \frac{P(N,K)}{K!} = \frac{N!}{(N-K)!\,K!}$$

	Selection Without Replacement	Selection With Replacement
Ordered Outcome (Sequence or String)	Permutations	Enumerations
Unordered Outcome (Set or Multiset)	Combinations	(We will not study this possibility...)

Finite Combinatorics

Combinations

$$\binom{N}{K} = \frac{P(N,K)}{K!} = \frac{N!}{(N-K)!\,K!}$$

Example:

Suppose we have 4 students, { A, B, C, D }, and we want to choose a team of 2 for a hackathon from among these 4. How many ways of doing this are there?

How many sequences of 2 from 4?

$$P(4,2) = \frac{4!}{(4-2)!} = \frac{24}{2} = 12$$

A B
A C
A D
B A
B C
B D
C A
C B
C D
D A
D B
D C

How many sets of 2 from 4?

$$\binom{N}{K} = \frac{N!}{(N-K)!\,K!} = \frac{4!}{(4-2)!\,2!} = \frac{24}{2*2} = \frac{24}{4} = 6$$

{ A, B }

{ A, C }

{ A, D }

{ B, C }

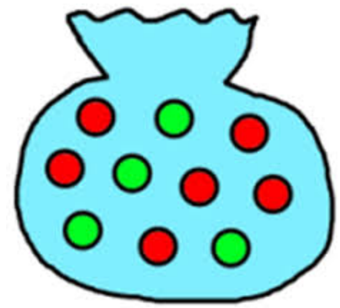
{ B, D }

{ C, D }

Unordering sequences of 2 students collapses 12 sequences to 12/2! = 6 sets.

Finite Combinatorics and Probabilities

Now let's consider a series of related problems involving permutations and combinations….



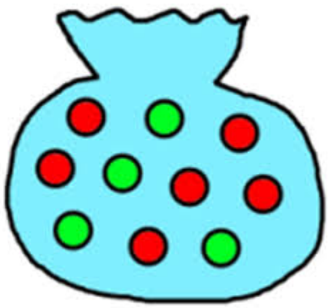
Suppose we have a bag containing 4 green balls and 6 red balls.

(A) Suppose 3 balls are drawn with replacement. What is the probability that you get the sequence [R, G, R]?

Recall:
with replacement =>
choices independent

Finite Combinatorics and Probabilities

Now let's consider a series of related problems involving permutations and combinations....



Suppose we have a bag containing 4 green balls and 6 red balls.

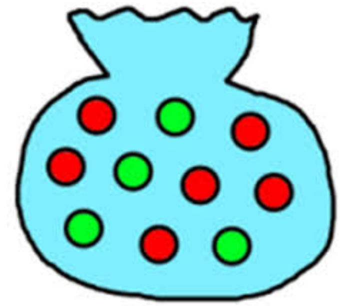
(A) Suppose 3 balls are drawn with replacement. What is the probability that you get the sequence [R, G, R]?

Solution: Since it with replacement, we have <u>independent choices</u>; we simply multiply the probabilities at each choice:

$$\frac{6}{10} * \frac{4}{10} * \frac{6}{10} = \frac{144}{1000} = 0.114$$

Note that ANY sequence of 2 R's and 1 G will have the same probability!

Finite Combinatorics and Probabilities

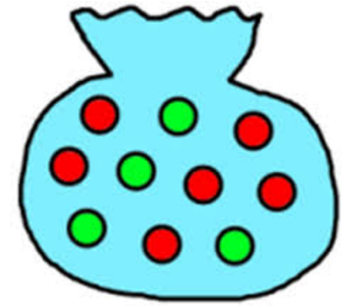


Suppose we have a bag containing 4 green balls and 6 red balls.

(B) Suppose 3 balls are drawn without replacement. What is the probability that you get the sequence [R, G, R]?

Recall:
without replacement =>
choices not independent

Finite Combinatorics and Probabilities

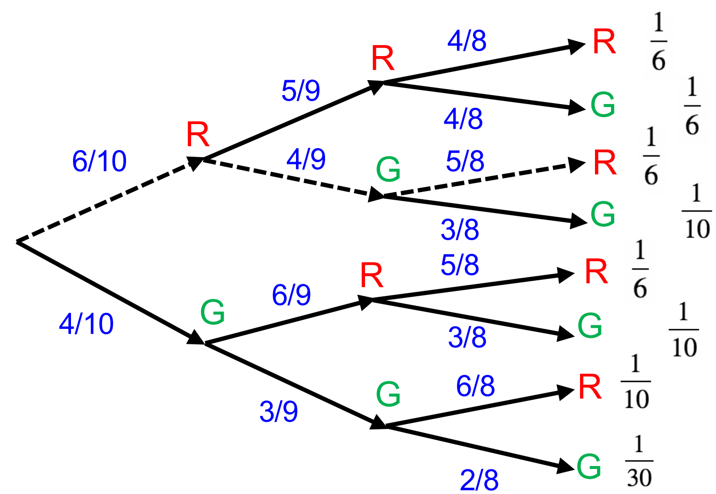


Suppose we have a bag containing 4 green balls and
6 red balls.

(B) Suppose 3 balls are drawn without replacement. What is the probability that you get the sequence [R, G, R]?

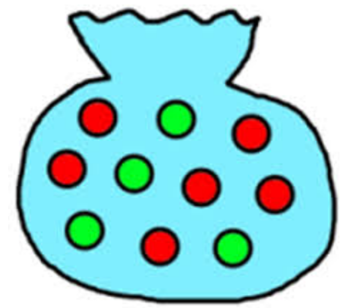
Solution: Since it without replacement, we calculate the (changing) probabilities at each stage:

$$\frac{6}{10} * \frac{4}{9} * \frac{5}{8} = \frac{120}{720} = \frac{1}{6}$$



Again, note that ANY sequence of 2 R's and 1 G will have the same probability!

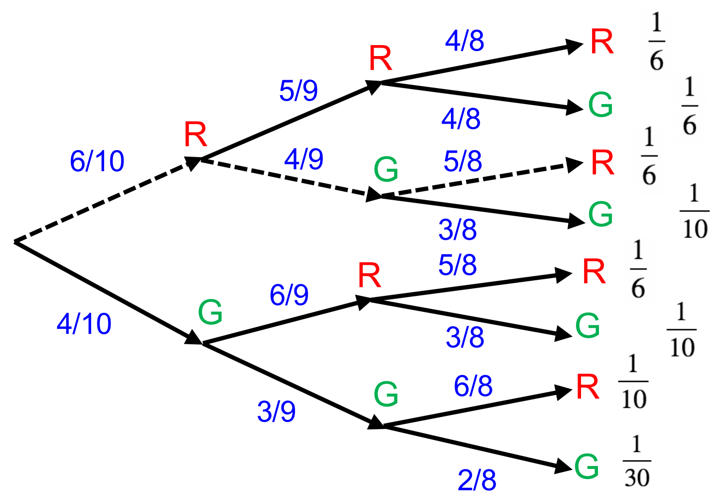
Finite Combinatorics and Probabilities



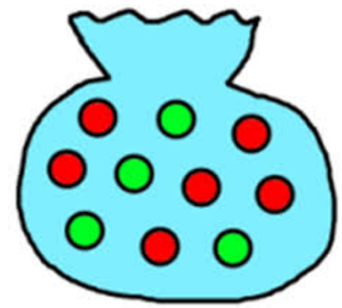
Suppose we have a bag containing 4 green balls and 6 red balls.

(C) Suppose 3 balls are drawn without replacement. What is the probability that you get a set { R, G, R }?

(Hint: First consider all the possible sequences….)



Finite Combinatorics and Probabilities

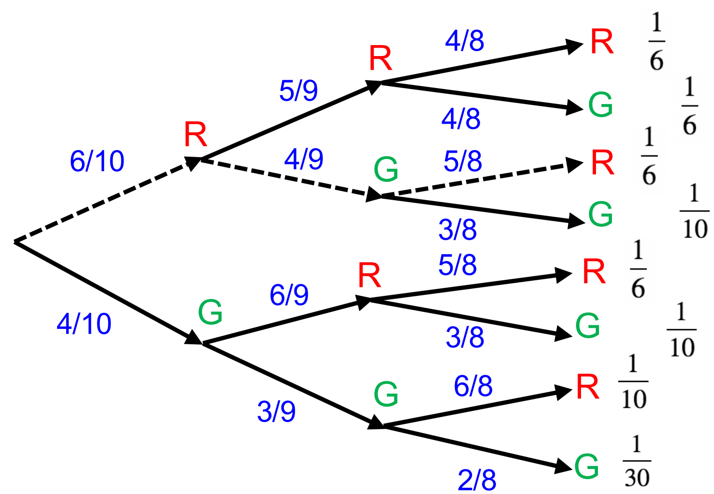


Suppose we have a bag containing 4 green balls and
6 red balls.

(C) Suppose 3 balls are drawn without replacement. What is the probability that you get a set { R, R, G }?

Solution 1: Adapt the solution to (B) (sequence =>unorder=> set):
there are three possible ways to get a sequence with 2 R's and 1 G:

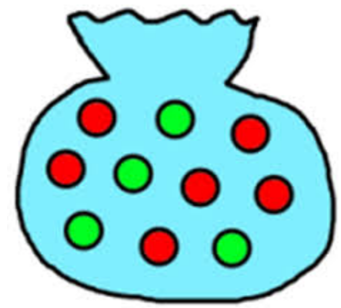
R R G
R G R
G R R



These are disjoint possibilities, so we can add! Probability of getting any of these three:

1/6 + 1/6 + 1/6 = 3/6 = 0.5

Finite Combinatorics and Probabilities



Suppose we have a bag containing 4 green balls and 6 red balls.

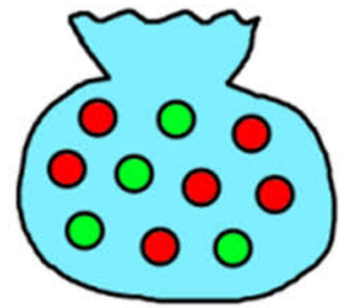
(C) Suppose 3 balls are drawn without replacement. What is the probability that you get a set { R, R, G }?

(Hint: Forget about sequences entirely, and consider how to do it using combinations, recalling that

$$P(A) = \frac{|A|}{|S|}$$

and think about how to choose the appropriate sets...)

Finite Combinatorics and Probabilities



Suppose we have a bag containing 4 green balls and 6 red balls.

(C) Suppose 3 balls are drawn without replacement. What is the probability that you get a set { R, R, G }? Let event A = "you get the set { R, R, G }."

Solution 2: How many possible ways of choosing a set of 3 from the 10 balls?

$$|S| = \binom{10}{3} = \frac{10!}{(10-3)!\,3!} = 120$$

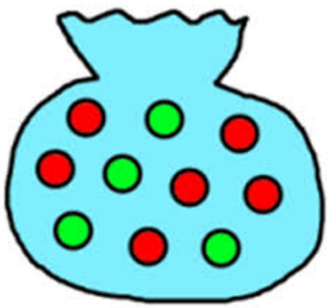
$$P(A) = \frac{|A|}{|S|}$$

How many ways of choosing a set of 3 balls with 2 R and 1 G?
Note: These choices are independent, so we can multiply:

$$|A| = \binom{6}{2}\binom{4}{1} = 15 * 4 = 60$$

So, $$P(A) = \frac{\binom{6}{2}\binom{4}{1}}{\binom{10}{3}} = 0.5$$

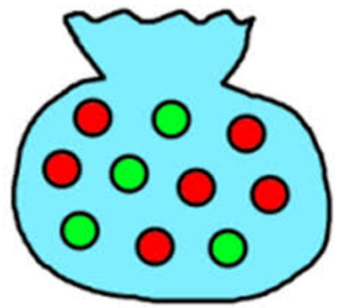
Finite Combinatorics and Probabilities



Suppose we have a bag containing 4 green balls and 6 red balls.

(D) Suppose 5 balls are drawn without replacement. What is the probability that you get 2 Red balls and 3 Green balls?

Finite Combinatorics and Probabilities



Suppose we have a bag containing 4 green balls and 6 red balls.

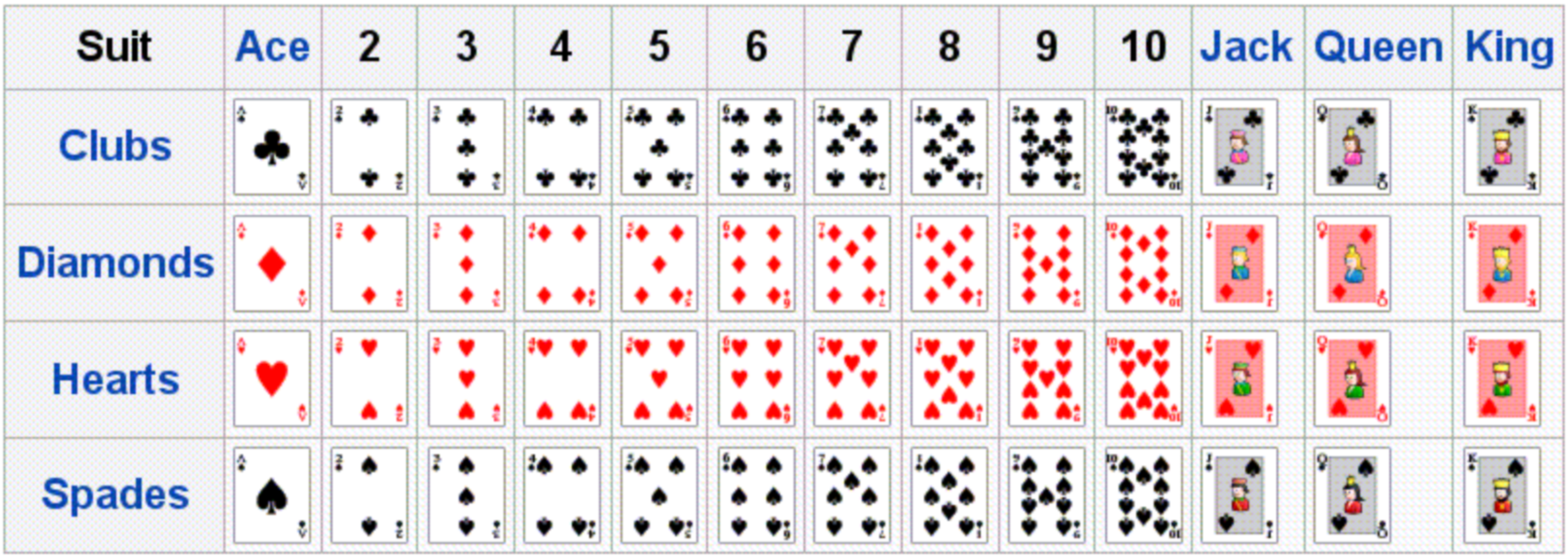
(D) Suppose 5 balls are drawn without replacement. What is the probability that you get 2 Red balls and 3 Green balls?

```
In [3]: from scipy.special import comb

        comb(6,2)*comb(4,3)/comb(10,5)

Out[3]: 0.23809523809523808
```

Finite Combinatorics and Probabilities

Poker Probabilities

Example set of 52 poker playing cards



A poker hand is 5 cards (a set) chosen without replacement.

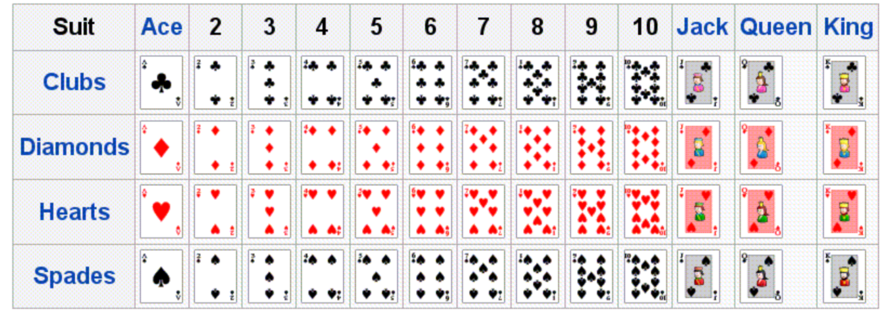
How many possible poker hands?

$$|S| = \binom{52}{5} = 2,598,960$$

Poker Probability

Poker hands are a great example of how to think about probability involving sets.

Example set of 52 poker playing cards



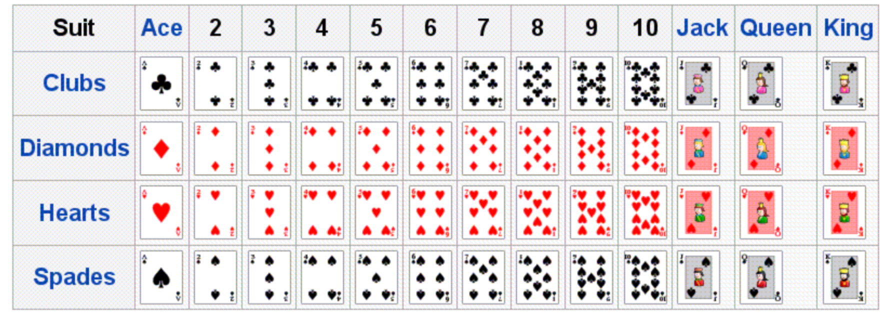
Example: What is the probability that you get a hand with 3 red cards and 2 black cards?

(Hint: Construct the hand by calculating how many such hands are possible, by constructing independent parts of the hand, and multiplying....)

Poker Probability

Poker hands are a great example of how to think about probability involving sets.

Example set of 52 poker playing cards

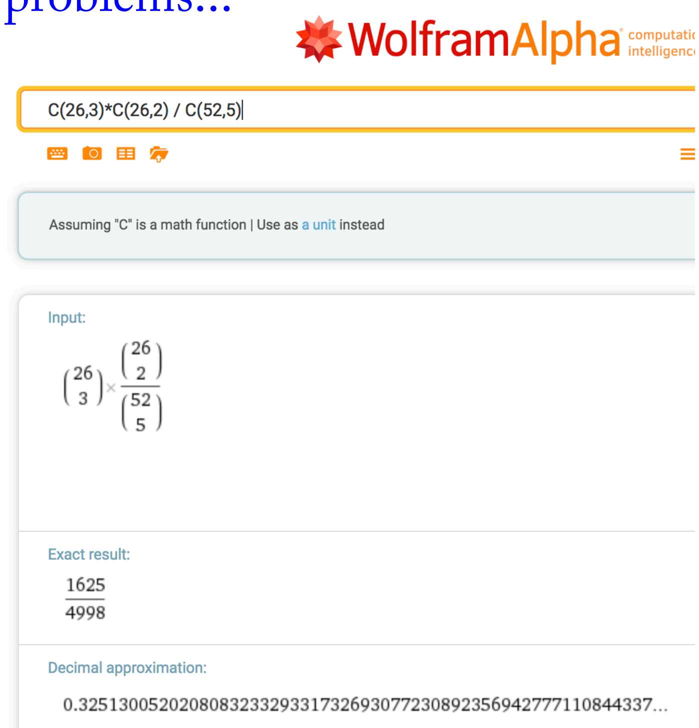


Example: What is the probability that you get a hand with 3 red cards and 2 black cards?

Solution:

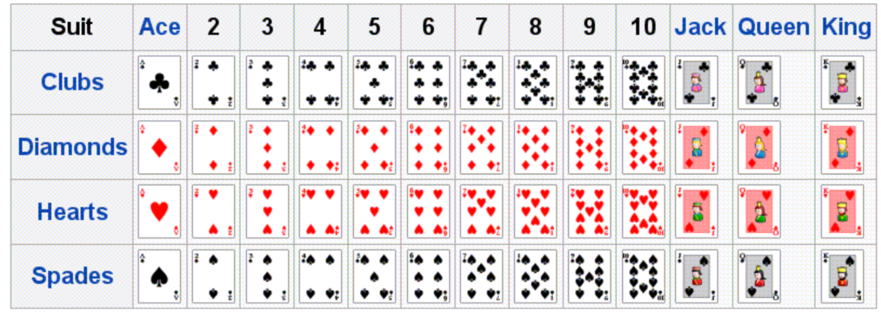
$$\frac{\binom{26}{3}\binom{26}{2}}{\binom{52}{5}} = 0.3251$$

By the way, Wolfram Alpha is the way to go when doing these problems...



Poker Probability

Example set of 52 poker playing cards



Poker hands are a great example of how to think about probability involving sets.

Example: What is the probability that you get a hand with 3 red cards and 2 black cards?

$$\frac{\binom{26}{3}\binom{26}{2}}{\binom{52}{5}} = 0.3251$$

Digression: Notice that you can also calculate this using **sequences** and **permutations**, but it is a bit more complicated, and you have more opportunities to get something wrong…

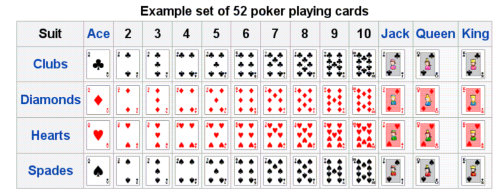
What is the probability of the exact sequence R R R B B ?

$$\frac{26}{52} * \frac{25}{51} * \frac{24}{50} * \frac{26}{49} * \frac{25}{48} = 0.0325$$

Now unorder it! How many permutations of this sequence of 5 symbols with duplicates?

$$\frac{5!}{3!\,2!} = \frac{120}{6 * 2} = 10$$

0.03251 * 10 = 0.3251



Poker Probability

Problem: What is the probability that a five-card hand has at least 3 Diamonds?

Solution: You need to separate this problem into cases, and might as well choose 3, 4, or 5 Diamonds, and for each find the probability and sum:

$$P(\text{ 3 Diamonds }) = \frac{\binom{13}{3}\binom{39}{2}}{\binom{52}{5}} = 0.0815$$

$$P(\text{ 4 Diamonds }) = \frac{\binom{13}{4}\binom{39}{1}}{\binom{52}{5}} = 0.0107$$

$$P(\text{ 5 Diamonds }) = \frac{\binom{13}{5}}{\binom{52}{5}} = 0.0005$$

These sum to 0.0928.

Poker Probability

Problem: What is the probability of a Flush (all the same suit)?

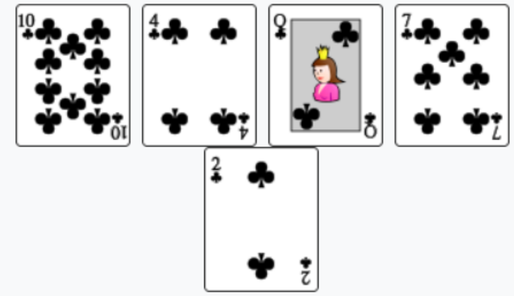
Solution: Choose a suit and then choose 5 cards from that suit:

$$\frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}} = 0.00198079$$

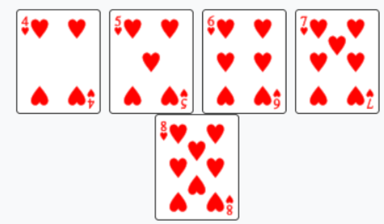
Note: This is the **cumulative** probability, in that it is a flush, but includes the straight and royal flushes. If we wish to exclude them, we must subtract all 40 of them:

$$\frac{\binom{4}{1}\binom{13}{5} - \binom{10}{1}\binom{4}{1}}{\binom{52}{5}} = 0.00196540$$

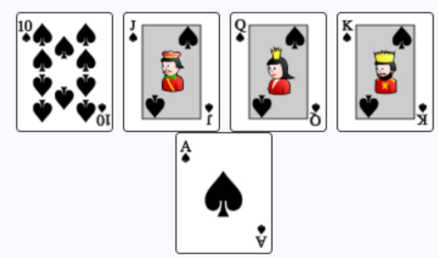
Flush (excluding royal flush and straight flush)



Straight flush (excluding royal flush)



Royal flush



Poker Probability

Problem: What is the probability of a Straight? Assume that Ace can be below 2 or above King.

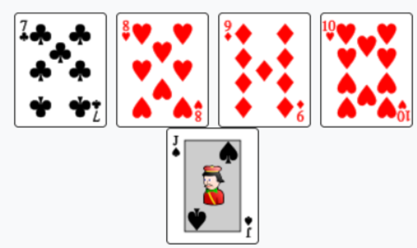
Solution: There are 10 sequences which form a straight, so just choose one of the 10 and then suits for each of the 5 cards:

```
A  2  3  4  5
2  3  4  5  6
3  4  5  6  7
4  5  6  7  8
5  6  7  8  9
6  7  8  9  10
7  8  9  10  J
8  9  10  J  Q
9  10  J  Q  K
10  J  Q  K  A
```

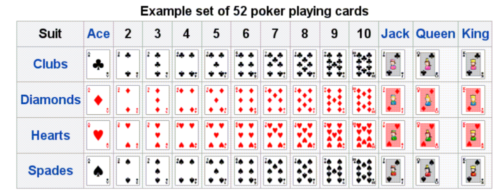
$$\frac{\binom{10}{1} * \binom{4}{1}^5}{\binom{52}{5}} = 0.00394$$

$$\frac{\binom{10}{1} * \binom{4}{1}^5 - \binom{10}{1}\binom{4}{1}}{\binom{52}{5}} = 0.003925$$

Straight (excluding royal flush and straight flush)

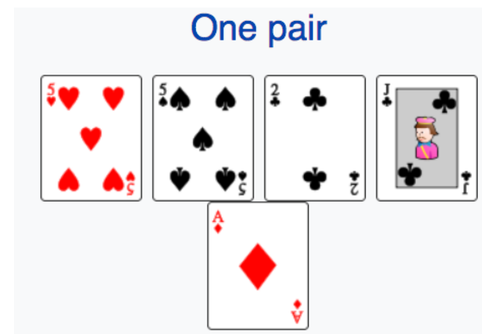


Poker Probability



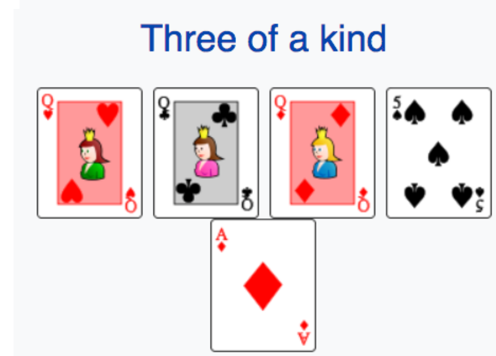
Problem: What is the probability of a Pair, 3-of-a-Kind, and 4-of-a-Kind?

Solution: First choose the denomination of the 2, 3 or 4 of a kind, then the suits of those cards, then the remaining cards of different denominations, again chosing the denomination, then the suits::



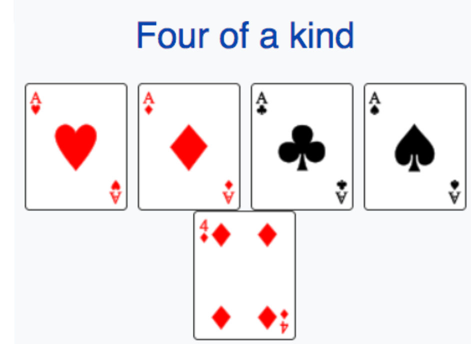
Pair:

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3}{\binom{52}{5}} = 0.4226$$



3-of-a-Kind:

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}} = 0.0211$$



4-of-a-Kind:

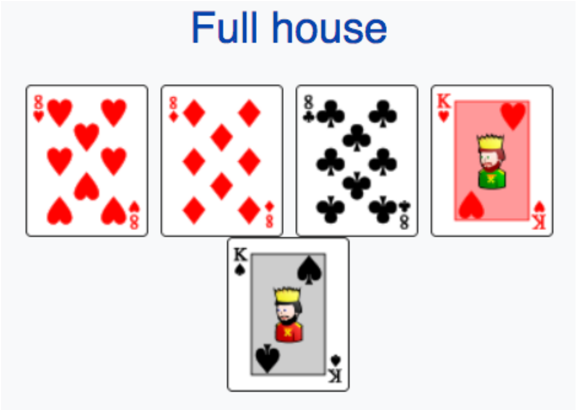
$$\frac{\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1}}{\binom{52}{5}} = 0.00024$$

Problem: Suppose in our class we have 160 students with 90 men and 70 women. I want to choose teams for an in-class demo with 5 men, 5 women, and also a scorekeeper (who can be anyone not on a team, man or woman). How many ways can I choose the teams and scorekeeper?

Solution: First choose the 5 men from the 90, then the 5 women from the 70, then one scorekeeper from the 150 people not on teams:

$$\binom{90}{5} * \binom{70}{5} * \binom{150}{1} = 79,787,790,884,062,800$$

Poker Probability



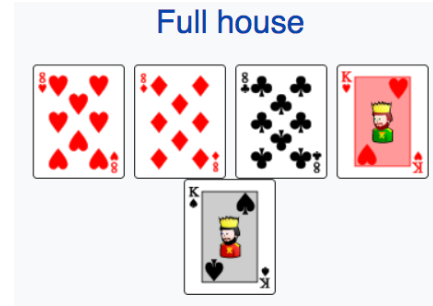
Problem: What is the probability of a Full House (3 of one denomination and 2 of another)?

Solution: First choose the denomination of the 3, then those 3 suits, then the denomination of the 2, then those 2 suits:

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} = 0.00144$$

We will consider one more hand, Two Pair, after some consideration of partitions.

Poker Probability



Question: Why is this not "choose the two ranks, then the suits for each"?

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}}{\binom{52}{5}} = 0.00144$$

$$\frac{\binom{13}{2}\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$$

It's the difference between sequences and sets!

Number of sets of 2 from 13: C(13, 2):

Number of sequences of 2 from 13, P(13, 2):

$$\binom{13}{2} = 78 \qquad P(13,2) = 13 \cdot 12 = 156$$

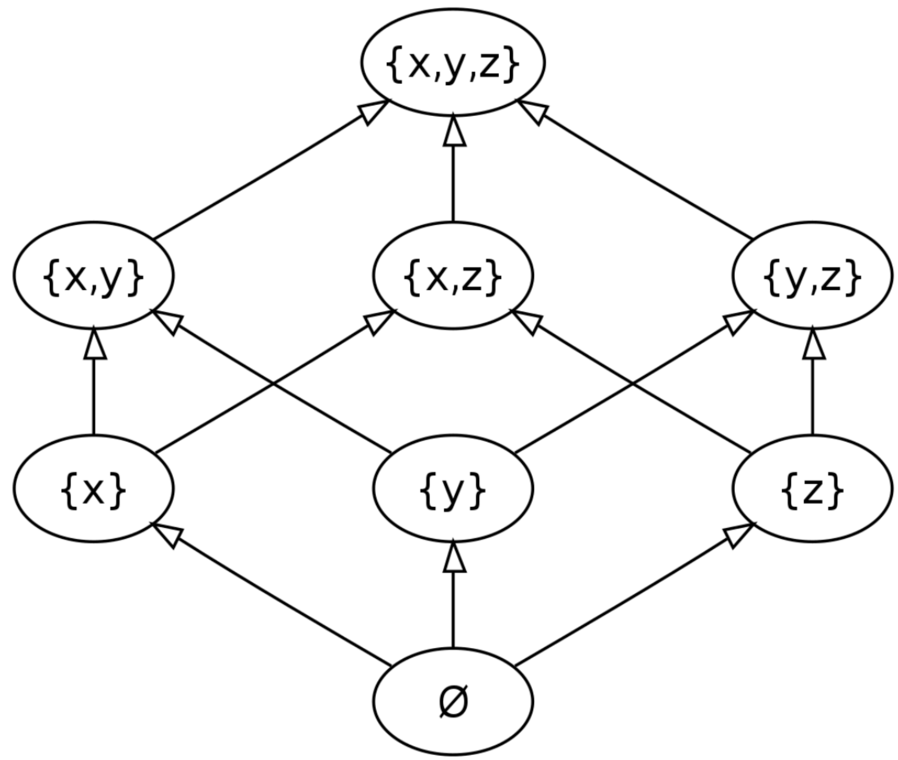
Counting Sets: Power Set

The Power Set of a set S is the set of all subsets (= set of all events):

$$\mathcal{P}(S) \quad =_{\text{def}} \quad \{ A \mid A \subseteq S \}$$

the cardinality of Power Set: $|\mathcal{P}(S)| = 2^{|S|}$

This is easy to see if we consider the **enumeration** of all sequences of { T, F } of length |S|, stating which elements of S are in the subset:



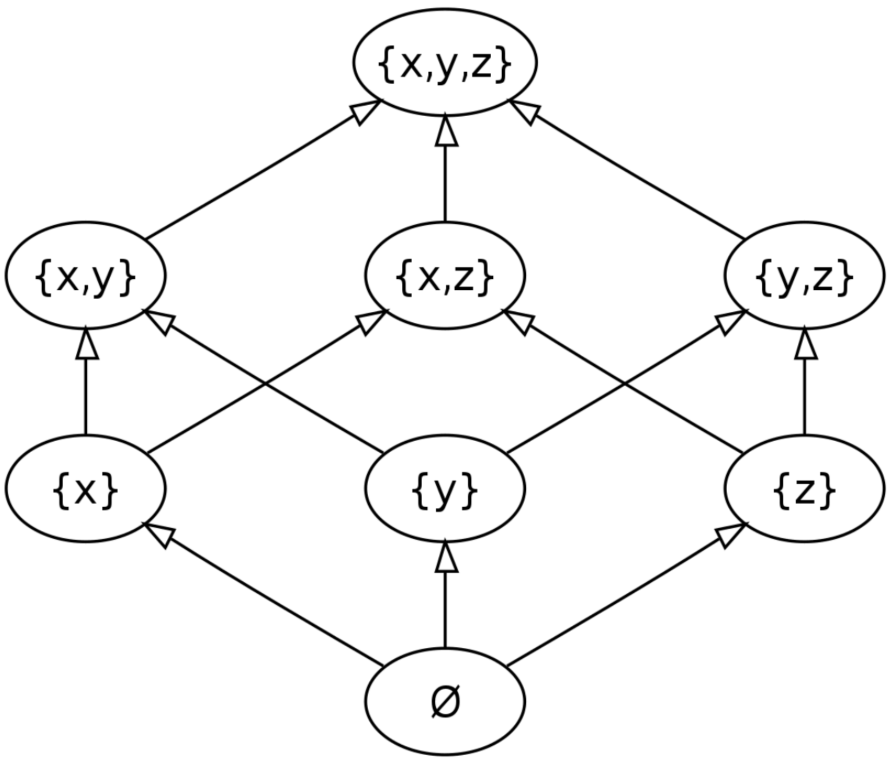

```
{ x, y }

x  y  z
T  T  F
```

$|\{T,F\}|^{|S|} = 2^{|S|}$

Counting Sets: Power Set and Combinations

There is of course a strong connection between the power set and combinations:

C(N, K) = how many subsets of size K from a set of size N.

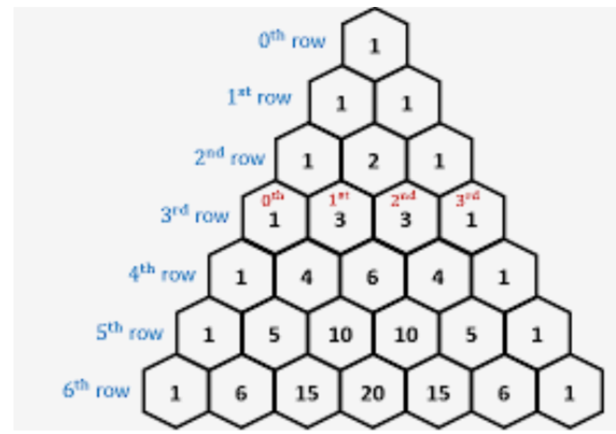


$$\binom{3}{3} = 1$$

$$\binom{3}{2} = 3$$

$$\binom{3}{1} = 3$$

$$\binom{3}{0} = 1$$



$$|\mathcal{P}(S)| = 2^{|S|} = \sum_{k=0}^{|S|} \binom{|S|}{k}$$

Counting Sets: Power Set and Combinations

Problem: A pizza shop claims they serve "more than 1000 kinds of pizza." You investigate and find they offer 10 different toppings (including cheese and tomato sauce among the 10). Is their claim correct? What about if we insist that a pizza must have cheese and tomato sauce at the very least?

Solution: Technically, yes, if you include all possible combinations of toppings, including cheese or no cheese and tomato sauce or no tomato sauce:

$$2^{10} = 1024$$

But this is a little funny, as it includes the empty set (no toppings, just bare crust!).

If you insist that "pizza" must have cheese and tomato sauce, then we have only

$$2^{8} = 256$$

Counting Sets: Partitions

A partition of a set S is a set of disjoint subsets which include every member of S:

$S = \{ 1, 2,3,4,5,6,7,8,9,10 \}$

Partitions:
$\{ \{1, 2,3,4\}, \{5, 6,7, 8, 9,10\} \}$
$\{ \{1\}, \{2,4\}, \{3,6,8\}, \{5, 7, 9\}, \{10\} \}$
$\{ \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\} \}$

Not :
$\{ \{1, 2,3,4\}, \{5, 6,7, 8, 9\} \}$ Doesn't have the 10 !
$\{ \{1, 2,3,4, 5\}, \{5, 6,7, 8, 9,10\} \}$ Not disjoint !

Counting Sets: Partitions

Counting partitions: always a good idea to try some examples first ...

Problem. Suppose we have five students { A, B, C, D, E }

We want to divide them into two teams of 3 and 2 people each. How many ways can we do this?

$$\binom{5}{3} = 10$$

Team of 3	Team of 2
A B C	D E
A B D	C E
A B E	C D
A C D	B E
A C E	B D
A D E	B C
B C D	B E
B C E	A D
B D E	A C
C D E	A B

Note: Once we have chosen the team of 3, the other team is determined!

For each one of these, there is only one set of 3 and set of 2, e.g.,

{ {A,B,C}, {D,E} }

Counting Sets: Partitions

Now let's try 2 teams of 2:

Problem. Suppose we have four students { A, B, C, D }

We want to divide them into two teams of 2 people each. How many ways can we do this? Is it this?

$$\binom{4}{2} = 6$$

Team of 2	Team of 2
A B	C D
A C	B D
A D	B C
B C	A D
B D	A C
C D	A B

Counting Sets: Partitions

Now let's try 2 teams of 2:

Problem. Suppose we have four students { A, B, C, D }

We want to divide them into two teams of 2 people each. How many ways can we do this?

$$\binom{4}{2} = 6$$

Not correct! We have overcounted by a factor of 2.

Team of 2	Team of 2
A B	C D
A C	B D
A D	B C
B C	A D
B D	A C
C D	A B

Same WAY of dividing into 2 teams!

As sets, these are the same way of building a partition:

{ {A,B}, {C,D} }   is same set of sets as:

{ {C,D}, {A,B} }

Counting Sets: Partitions

Now let's try 3 teams of 2:

Problem. Suppose we have six students { A, B, C, D, E, F }

We want to divide them into 3 teams of 2 people each. How many ways can we do this?

$$\binom{6}{2}\binom{4}{2} = 15 * 6 = 90$$

Team of 2	Team of 2	Team of 2
A B	C D	E F
A B	E F	C D
C D	A B	E F
C D	E F	A B
E F	A B	C D
E F	C D	A B

Note: Once we have chosen the first 2 teams of 2, the last team is determined!

All these "ways" of dividing into 3 teams of equal size are the same!

Overcounting by P(3,3) = 3!, correct answer is:

$$\frac{\binom{6}{2}\binom{4}{2}}{3!} = \frac{90}{6} = 15$$

The **Unordering Principle** strikes again!

Counting Sets: Partitions

Problem. Suppose we have 15 students and want to divide them into

2 teams of 3,
4 teams of 2, and
a single student who will be referee.

How many ways of doing this are there?

Solution: Use multi-nomial coefficients to remove the duplicates among teams you can't distinguish by size:

$$\frac{\binom{15}{3}\binom{12}{3}\binom{9}{2}\binom{7}{2}\binom{5}{2}\binom{3}{2}}{2! * 4!} = \frac{2,270,268,000}{48} = 47,297,250$$

Counting Sets: Partitions

Now suppose we distinguish the teams by NAME.

Problem. Suppose we have four students { A, B, C, D }

We want to divide them into two teams of 2 people each called "Attackers" and "Defenders." How many ways can we do this?

$$\binom{4}{2} = 6$$

Attackers	Defenders
A B	C D
A C	B D
A D	B C
B C	A D
B D	A C
C D	A B

Now there is no overcounting! Switching attackers and defenders gives you a different **way**. There are no **duplicate** ways.

This may seem obscure, but think about experiments involving a "test group" (who take a new drug) and a "control group" (who take a placebo). Switching the groups makes a difference!

Counting Sets: Partitions

Problem. Suppose we have 15 students and want to divide them into

-- 2 teams of 3, named “MIT Attackers” and “Harvard Attackers”
-- 4 teams of 2, all defenders (all unnamed); and
-- a single student who will be referee.

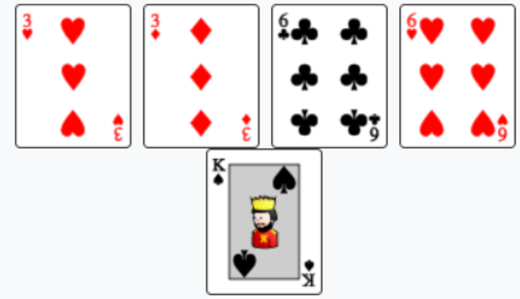
How many ways of doing this are there?

Solution: Use multinomial coefficients to remove the duplicates among teams you can’t distinguish by size or name:

$$\frac{\binom{15}{3}\binom{12}{3}\binom{9}{2}\binom{7}{2}\binom{5}{2}\binom{3}{2}}{4!} = \frac{2,270,268,000}{24} = 94,594,500$$

Poker Probability -- One Last Time:

Two pair



Problem: What is the probability of Two Pair (2 of one denomination and 2 of different denominations)?

Solution: First choose the denomination of the first pair, then those 2 suits, then the denomination of the second pair, then those 2 suits, then the remaining card:

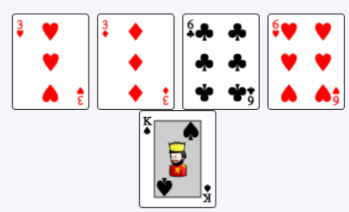
$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2}\binom{11}{1}\binom{4}{1}}{\binom{52}{5}} = 0.09508$$

But wait.... This doesn't correspond to the web page OR our experiments, which seem to suggest it is too high by a factor of 2. What is wrong?

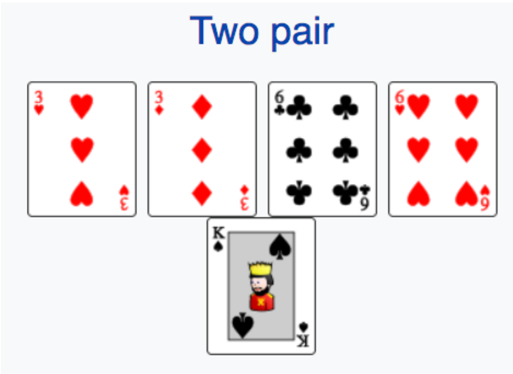
Problem 5 (C): What is probability of Two Pairs in Poker?

```
In [98]: # probability should be close to analytical value of 0.047539

         seed(0)
         num_trials = 10**5

         print('Probability of two pairs in poker is ' + str(probability_of
```

```
Probability of two pairs in poker is 0.04618
```

Two pair 	858	123,552	4.7539%

Poker Probability One Last Time:



Just another example of you-know-what, in this case, overcounting the two pairs:

2D 2H 3C 3D 5S

3C 3D 2D 2H 5S

These are the same hand, but would be counted twice!

Same problem as: you have 52 students, and want to select 2 teams of 2, plus a referee.

So we could divide by 2! to get the right number:

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2}\binom{11}{1}\binom{4}{1}}{2 * \binom{52}{5}} = 0.04754$$

OR we could choose a set of 2 ranks to get the two pairs:

$$\frac{\binom{13}{2}\binom{4}{2}^2\binom{44}{1}}{\binom{52}{5}}$$